

Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method

Yves Brihaye

*Physique-Mathématique, Université de Mons-Hainaut,
Mons, Belgium
E-mail: brihaye@umh.ac.be*

Eugen Radu

*Laboratoire de Mathématiques et Physique Théorique, Université François-Rabelais,
Tours, France
E-mail: radu@lmpt.univ-tours.fr*

ABSTRACT: We propose to compute the action and global charges of the asymptotically anti-de Sitter solutions in Einstein-Gauss-Bonnet theory by adding boundary counterterms to the gravitational action. The general expression of the counterterms and the boundary stress tensor is presented for spacetimes of dimension $d \leq 9$. We apply this technique for several different types of black objects. Apart from static and rotating black holes, we consider also Einstein-Gauss-Bonnet black string solutions with negative cosmological constant.

KEYWORDS: Classical Theories of Gravity, Black Holes.

Contents

1. Introduction	1
2. The formalism	3
2.1 The action and field equations	3
2.2 The counterterms and boundary stress tensor	4
3. EGB-AdS black holes	8
3.1 Static solutions	8
3.2 Rotating black holes with equal magnitude angular momenta	10
4. EGB-AdS black string solutions	12
4.1 The metric ansatz and reduced action	13
4.2 The asymptotics	13
4.3 The conserved charges and entropy	15
4.4 Numerical results	17
5. Further remarks	22

1. Introduction

The AdS/CFT (anti-de Sitter space/conformal field theory) correspondence [1] attracted a lot of attention in the last decade. This conjecture tells that the partition function in a $d - 1$ dimensional CFT is given in terms of the classical action in a d -dimensional gravity theory with negative cosmological constant.

In light of this correspondence, asymptotically AdS black objects would offer the possibility of studying some aspects of the nonperturbative structure of certain quantum field theories. Therefore it is of interest to find more general gravity solutions with negative cosmological constant and to study their physics, trying to relate it to the physics of the boundary theory.

An interesting case is provided by the solutions of the Einstein-Gauss-Bonnet (EGB) theory in $d \geq 5$ dimensions, since the Gauss-Bonnet (GB) term appears as the first curvature stringy correction to general relativity [2, 3], when assuming that the tension of a string is large as compared to the energy scale of other variables. The EGB equations contain no higher derivatives of the metric tensor than second order and it has proven to be free of ghost when expanding around flat space. In the AdS/CFT correspondence, the

introduction of such higher order terms¹ corresponds to next to leading order corrections to the $1/N$ expansion of the CFT [5–7].

Similar to the case of Einstein gravity, when computing quantities like the action and mass of EGB solutions, one encounters infrared divergences, associated with the infinite volume of the spacetime manifold. The traditional approach to this problem is to use a background subtraction whose asymptotic geometry matches that of the solutions. However, such a procedure causes the resulting physical quantities to depend on the choice of reference background; furthermore, it is not possible in general to embed the boundary surface into a background spacetime. For asymptotically AdS solutions of the Einstein gravity, one can instead deal with these divergences via the counterterm method inspired by the AdS/CFT correspondence [8]. The procedure consists of adding to the action suitable boundary counterterms, which are built up with curvature invariants of the boundary metric and thus obviously they do not alter the bulk equations of motion. This yields a well-defined Brown-York boundary stress tensor [9] and a finite action and mass of the system.²

In principle, there are no obstacles in computing the action and global charges of EGB solutions by using a similar approach. At any given dimension one can write down only a finite number of counterterms that do not vanish at infinity and this does not depend upon the bulk theory is Einstein or GB. However, the presence in this case of a new length scale implies a complicated expression for the coefficients of the boundary counterterms and makes the procedure technically much more involved. To our knowledge, the only cases considered in the literature³ correspond to $d = 5$ solutions [15–17], and solutions in arbitrary dimension possessing a zero curvature⁴ boundary [18].

The first aim of this paper is to generalize the boundary counterterms and the quasilocal stress energy tensor in [8] to EGB theory. Our results are valid for configurations with $d \leq 9$, although a general counterterm expression is also conjectured. Upon application of the Gibbs-Duhem relation to the partition function, this yields an expression for the entropy of a black objects which contains the GB corrections.

In the second part of this paper we apply this general formalism to asymptotically AdS black holes and black strings in EGB theory. In the static black hole case, where a closed form solution is known, the expressions we find for the mass-energy and entropy agree with known results in the literature. Apart from these static solutions, we consider also rotating black holes with equal magnitude angular momenta in a odd number of spacetime dimensions. The second type of black objects studied in this work are the EGB generalization of the asymptotically AdS black strings considered in [19, 20]. We find that these solutions

¹Note, however, that the first non vanishing corrections that appear in the type IIB string effective action differ from EGB, involving eight derivatives, *i.e.* a term involving four powers of the Riemann tensor together with its supersymmetric counterparts [2, 4].

²This approach has been generalized for spacetimes which are not asymptotically AdS and the cosmological constant is replaced by a dilaton scalar potential, see *e.g.* [10]

³An alternative regularization prescription for any Lovelock theory with AdS asymptotics has been proposed in [11–13]. This approach uses boundary terms with explicit dependence on the extrinsic curvature K_{ab} , also known as Kounterterms [14].

⁴In this case all curvature invariants are zero except for a constant. Therefore the only possible boundary counterterm is proportional to the volume of the boundary, regardless of the number of dimensions.

present some new qualitative features as compared to the case of Einstein gravity.

Our paper is structured as follows: in the next section we explain the model and describe the computation of the physical quantities of the solutions such as their action and mass-energy. Our proposal for the general counterterm expression in EGB theory is also presented there. The general properties of the black hole solutions are presented in section 3, while in section 4 we present the new results obtained in the case of EGB black strings. Their construction is based on both analytical and numerical techniques. We give our conclusions and remarks in the final section.

2. The formalism

2.1 The action and field equations

We consider the EGB action with a negative cosmological constant $\Lambda = -(d-2)(d-1)/2\ell^2$

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^d x \sqrt{-g} \left(R - 2\Lambda + \frac{\alpha}{4} L_{\text{GB}} \right), \quad (2.1)$$

where R is the Ricci scalar and

$$L_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}, \quad (2.2)$$

is the GB term. In four dimensions this is a topological invariant; in higher dimensions it is the most general quadratic expression which preserves the property that the equations of motion involve only second order derivatives of the metric. L_{GB} can also be viewed as the second order term in the Lovelock theory of gravity constructed from vielbein, the spin connection and their exterior derivatives without using the Hodge dual, such that the field equations are second order [21, 22]. The constant α in (2.1) is the GB coefficient with dimension $(length)^2$ and is positive in the string theory. We shall therefore restrict in this work to the case $\alpha > 0$, although the counterterm expression does not depend on this choice.

The variation of the action (2.1) with respect to the metric tensor results in the equations of the model

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{\alpha}{4}H_{\mu\nu} = 0, \quad (2.3)$$

where

$$H_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_{\nu}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^{\sigma}_{\nu} + RR_{\mu\nu}) - \frac{1}{2}L_{\text{GB}}g_{\mu\nu}. \quad (2.4)$$

For a well-defined variational principle, one has to supplement the action (2.1) with the Gibbons-Hawking surface term [23]

$$I_b^{(E)} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} K, \quad (2.5)$$

and its counterpart for the GB gravity [3]

$$I_b^{(GB)} = -\frac{\alpha}{16\pi G} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} \left(J - 2G_{ab}K^{ab} \right), \quad (2.6)$$

where γ_{ab} is the induced metric on the boundary, K is the trace of the extrinsic curvature of the boundary, G_{ab} is the Einstein tensor of the metric γ_{ab} and J is the trace of the tensor

$$J_{ab} = \frac{1}{3}(2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}) . \quad (2.7)$$

We shall consider spacetimes of negative constant curvature in the asymptotic region, which implies the asymptotic expression of the Riemann tensor $R_{\mu\nu}{}^{\lambda\sigma} = -(\delta_\mu^\lambda\delta_\nu^\sigma - \delta_\mu^\sigma\delta_\nu^\lambda)/\ell_c^2$, where ℓ_c is the new effective radius of the AdS space in EGB theory.⁵ We have found convenient to write

$$\ell_c = \ell\sqrt{\frac{1+U}{2}}, \quad \text{with} \quad U = \sqrt{1 - \frac{\alpha(d-3)(d-4)}{\ell^2}}, \quad (2.8)$$

which results in a compact form for the counterterms and a simpler expression of the black string asymptotics. One should also notice the existence of an upper bound for the GB coefficient, $\alpha \leq \alpha_{\max} = \ell^2/(d-3)(d-4)$, which holds for all asymptotically AdS solutions.

2.2 The counterterms and boundary stress tensor

The action and global charges of the EGB- Λ solutions are computed by using a suitable generalization of the procedure proposed by Balasubramanian and Kraus [8] for Einstein gravity with negative cosmological constant. This technique was inspired by the AdS/CFT correspondence (since quantum field theories in general contains counterterms) and consists of adding to the action suitable boundary counterterms I_{ct} , which are functionals only of curvature invariants of the induced metric on the boundary. The number of terms that appears grows with the dimension of the spacetime.

To regularize the action of the $d < 8$ solutions, we supplement the general action (which contains the surface terms for Einstein and GB gravity) with the following boundary counterterms

$$I_{\text{ct}}^0 = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} \left\{ - \left(\frac{d-2}{\ell_c} \right) \left(\frac{2+U}{3} \right) - \frac{\ell_c \Theta(d-4)}{2(d-3)} (2-U) \mathbb{R} \right. \quad (2.9)$$

$$\left. - \frac{\ell_c^3 \Theta(d-6)}{2(d-3)^2(d-5)} \left[U \left(\mathbb{R}_{ab} \mathbb{R}^{ab} - \frac{d-1}{4(d-2)} \mathbb{R}^2 \right) - \frac{d-3}{2(d-4)} (U-1) L_{\text{GB}} \right] \right\},$$

where \mathbb{R} , \mathbb{R}^{ab} and L_{GB} are the curvature, the Ricci tensor and the GB term associated with the induced metric γ . Also, $\Theta(x)$ is the step-function with $\Theta(x) = 1$ provided $x \geq 0$, and zero otherwise. One can see that as $\alpha \rightarrow 0$ ($U \rightarrow 1$), one recovers the known counterterm expression in the Einstein gravity [8, 25, 26].

A similar expression can be written for higher dimension than seven, with new terms entering at any even d . However, their complexity strongly increases with d . Based on the relations we have derived for $d < 10$, we conjecture the following general expression for the

⁵For a precise definition of asymptotically AdS spacetime in higher curvature gravitational theories, see *e.g.* [24].

boundary counterterm in the EGB theory⁶

$$I_{\text{ct}}^0 = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-\gamma} \left\{ \sum_{k \geq 1} \Theta(d-2k) \left(f_1(U) L_E + f_2(U) \mathcal{L}_{(k-1)} \right) \right\}, \quad (2.10)$$

where L_E is the corresponding k -th part of the counterterm lagrangean for a theory with only Einstein gravity in the bulk (with the length scale ℓ in front of it replaced by the new effective AdS radius ℓ_c) and $\mathcal{L}_{(k-1)}$ is the $(k-1)$ term in the Lovelock hierarchy. The functions $f_1(U)$, $f_2(U)$ have the general expression

$$f_1 = 1 + c_1(U-1), \quad f_2 = \ell_c^{2k-3} c_2(U-1), \quad (2.11)$$

where c_1, c_2 are d -dependent coefficients. The series truncates for any fixed dimension, with new terms entering at every new even value of d .

The relation (2.9) contains already the cases $k = 1, 2, 3$. For $k = 1$ one finds $c_1 = 0$, $c_2 = -(d-2)/3$, while $L_E = -(d-2)/\ell_c$, $\mathcal{L}_{(0)} = 1$. Taking $k = 2$ yields $c_1 = 0$, $c_2 = 1/(2(d-3))$ and $L_E = -\ell_c R/2(d-3)$, $\mathcal{L}_{(1)} = R$. The value $k = 3$ implies the following expression for the new terms in (2.10)

$$c_1 = 1, \quad c_2 = \frac{1}{4(d-3)(d-4)(d-5)}, \quad (2.12)$$

$$L_E = -\frac{\ell_c^3}{2(d-3)^2(d-5)} \left(R_{ab} R^{ab} - \frac{d-1}{4(d-2)} R^2 \right), \quad \mathcal{L}_{(2)} = L_{\text{GB}}.$$

The case $k = 4$ is more involved, with

$$L_E = \frac{\ell_c^5}{(d-3)^3(d-5)(d-7)} \left(\frac{3d-1}{4(d-2)} R R^{ab} R_{ab} - \frac{(d-1)(d+1)}{16(d-2)^2} R^3 \right. \\ \left. - 2R^{ab} R^{cd} R_{abcd} + \frac{d-3}{2(d-2)} R^{ab} \nabla_a \nabla_b R - R^{ab} \nabla^2 R_{ab} + \frac{1}{2(d-2)} R \nabla^2 R \right), \quad (2.13)$$

as given in [27], and the third Lovelock term

$$\mathcal{L}_{(3)} = 2R^{abcd} R_{cdef} R^{ef}_{ab} + 8R^{ab}_{cd} R^{ce}_{bf} R^{df}_{ae} + 24R^{abcd} R_{cdbe} R_a^e \\ + 3R R^{abcd} R_{cdab} + 24R^{abcd} R_{cd} R_{db} + 16R^{ab} R_{bc} R_a^c - 12R R^{ab} R_{ab} + R^3, \quad (2.14)$$

the constants c_1, c_2 in the expression of f_1, f_2 being

$$c_1 = 31/30, \quad c_2 = -19/57600, \quad \text{for } d = 8, \quad (2.15)$$

$$c_1 = 2365/2313, \quad c_2 = -149/2664576, \quad \text{for } d = 9.$$

⁶Following [8, 25], the $d \leq 9$ counterterms were obtained by demanding cancellation of divergencies for a number of solutions in EGB theory. For example, for $d \leq 7$, the only geometric terms that possible do not vanish at infinity are $1/\ell_c$, R and R^2 , $R_{ab} R^{ab}$, $R_{abcd} R^{abcd}$. However, the last term does not appear in the Einstein gravity expression. Therefore, we have found convenient to express the $d \leq 7$ counterterm (2.9) as the sum of a Einstein-gravity part (multiplied with some U -dependent factors) and a GB term with a factor of $U-1$ in front of it (*i.e.* it vanishes in the Einstein gravity limit of the bulk theory). The same approach yields a relatively simple expression of the counterterm also for $d = 8, 9$. For the generality of the results proposed here, see the comments in the last section of this work.

Varying the total action with respect to the boundary metric γ_{ab}

$$T_{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma^{ab}} \left(I + I_b^{(E)} + I_b^{(GB)} + I_{ct}^0 \right) \quad (2.16)$$

results in the boundary stress-energy tensor:

$$\begin{aligned} T_{ab} = & K_{ab} - \gamma_{ab} K + \frac{\alpha}{2} \left(Q_{ab} - \frac{1}{3} Q \gamma_{ab} \right) - \frac{d-2}{\ell_c} \gamma_{ab} \left(\frac{2+U}{3} \right) + \frac{\ell_c \Theta (d-4)}{d-3} (2-U) \left(R_{ab} - \frac{1}{2} \gamma_{ab} R \right) \\ & + \ell_c^3 \Theta (d-6) \left\{ \frac{U}{(d-3)^2 (d-5)} \left(-\frac{1}{2} \gamma_{ab} \left(R_{cd} R^{cd} - \frac{(d-1)}{4(d-2)} R^2 \right) \right. \right. \\ & \left. \left. - \frac{(d-1)}{2(d-2)} R R_{ab} + 2 R^{cd} R_{cabd} - \frac{d-3}{2(d-2)} \nabla_a \nabla_b R + \nabla^2 R_{ab} - \frac{1}{2(d-2)} \gamma_{ab} \nabla^2 R \right) \right. \\ & \left. - \frac{U-1}{2(d-3)(d-4)(d-5)} H_{ab} \right\} + \dots, \end{aligned} \quad (2.17)$$

where [28]

$$\begin{aligned} Q_{ab} = & 2K K_{ac} K_b^c - 2K_{ac} K^{cd} K_{db} + K_{ab} (K_{cd} K^{cd} - K^2) \\ & + 2K R_{ab} + R K_{ab} - 2K^{cd} R_{cabd} - 4R_{ac} K_b^c, \end{aligned} \quad (2.18)$$

and H_{ab} given by (2.4) in terms of the boundary metric γ_{ab} .

The expression (2.17) is valid for $d \leq 7$. For $d \geq 8$, the boundary stress tensor receives a supplementary contribution from the $k = 4$ term in (2.10). The Einstein part of it can be found in ref. [27]; the other contribution represents the third Lovelock tensor as derived in ref. [29] (from (2.10), (2.16) they are both multiplied with U -dependent factors). These extra-terms are very long and we prefer do not include them here.

However, for odd values of d , the counterterms proposed above may fail to regularize the action. This is the case of the black string solutions we shall discuss in the section 4, or of the EGB Euclidean AdS_d metric $ds^2 = \frac{dr^2}{1+r^2/\ell_c^2} + r^2 d\Omega_{d-1}^2$ (where $d\Omega^2$ is the unit metric on the sphere). Already in the $\alpha = 0$ case, the action of these solutions presents a logarithmic divergence, whose coefficient is related to the conformal Weyl anomaly in the dual theory defined in a even dimensional spacetime.

However, the divergences are removed by adding the following extra term to (2.1)

$$\begin{aligned} I_{ct}^s = & \frac{1}{8\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{-\gamma} \log \left(\frac{r}{\ell_c} \right) \left\{ \delta_{d,5} \frac{\ell_c^3}{8} \left[U \left(\frac{1}{3} R^2 - R_{ab} R^{ab} \right) - (U-1) (R^2 \right. \right. \\ & \left. \left. - 4R_{ab} R^{ab} + R_{abcd} R^{abcd} \right] - \delta_{d,7} \frac{\ell_c^5}{128} \left[\frac{1}{18} (19U-1) \left(R R^{ab} R_{ab} - \frac{3}{25} R^3 - 2R^{ab} R^{cd} R_{cabd} \right) \right. \right. \\ & \left. \left. - \frac{1}{10} R^{ab} \nabla_a \nabla_b R + R^{ab} \square R_{ab} - \frac{1}{10} R \square R \right) - \frac{11}{54} (U-1) \mathcal{L}_{(2)} \right] + \dots \left. \right\}, \end{aligned} \quad (2.19)$$

(with r the radial coordinate normal to the boundary). This gives a supplementary contribution to the boundary stress tensor which removes the logarithmic divergencies from the solutions' global charges. Note that a logarithmic contribution to the counterterms also appears naturally in a formulation of the holographic renormalization procedure in terms of the extrinsic curvature [30].

The computation of the global charges associated with the Killing symmetries of the boundary metric is done in a similar way to the case of Einstein gravity. One assumes that the boundary submanifold can be foliated in a standard ADM form

$$\gamma_{ab}dx^a dx^b = -N^2 dt^2 + \sigma_{ij}(dy^i + N^i dt)(dy^j + N^j dt), \quad (2.20)$$

where N and N^i are the lapse function, respectively the shift vector, and y^i , $i = 1, \dots, d-2$ are the intrinsic coordinates on a closed surface Σ of constant time t on the boundary. Then a conserved charge

$$\Omega_\xi = \oint_\Sigma d^{d-2}y \sqrt{\sigma} u^a \xi^b T_{ab}, \quad (2.21)$$

can be associated with the closed surface Σ (with normal u^a), provided the boundary geometry has an isometry generated by a Killing vector ξ^a . For example, the conserved mass/energy M is the charge associated with the time translation symmetry, with $\xi = \partial/\partial t$.

The background metric upon which the dual field theory resides is $h_{ab} = \lim_{r \rightarrow \infty} \frac{\ell_p^2}{r^2} \gamma_{ab}$. The expectation value of the stress tensor of the dual theory can be computed using the relation [31]:

$$\sqrt{-h} h^{ab} \langle \tau_{bc} \rangle = \lim_{r \rightarrow \infty} \sqrt{-\gamma} \gamma^{ab} T_{bc}, \quad (2.22)$$

and is expected to present a nontrivial dependence on the parameter α .

The Hawking temperature T_H of the black objects is found by demanding regularity of the Euclideanized manifold, or equivalently, by evaluating the surface gravity. The entropy of the black hole objects is computed in this work using the (Euclidean) path-integral formalism.⁷ In this approach, the gravitational thermodynamics is based on the general relation [34]

$$Z = \int D[g] D[\Psi] e^{-I[g, \Psi]} \simeq e^{-I_{cl}},$$

where $D[g]$ is a measure on the space of metrics g , $D[\Psi]$ a measure on the space of matter fields Ψ and I_{cl} is the classical action evaluated on the equations of motion of the gravity/matter system. This yields an expression for the entropy (with $\beta = 1/T_H$)

$$S = \beta(M - \mu_i \mathfrak{C}_i) - I_{cl}, \quad (2.23)$$

upon application of the Gibbs-Duhem relation to the partition function, with chemical potentials \mathfrak{C}_i and conserved charges μ_i (see *e.g.* [35]). In Einstein gravity, the entropy computed in this way is one quarter of the event horizon area for any black object. However, a GB term in the action may provide a nonzero contribution to S . Finding this

⁷Note, however, that not all solutions with Lorentzian signature present reasonable Euclidean counterparts (see, *e.g.* [32]), in which case one is forced again to consider a 'quasi-Euclidean' approach as described in [33]. In this case the action I is regarded as a functional over complex metrics that are obtained from the real, stationary, Lorentzian metrics by using a transformation that mimics the effect of the Wick rotation $t \rightarrow i\tau$. The values of the extensive variables of the complex metric that extremize the path integral are the same as the values of these variables corresponding to the initial Lorentzian metric.

correction within the Euclidean approach requires a consideration of concrete configurations, no general result being available. However, all solutions should satisfy the first law of thermodynamics

$$dS = \beta(dM - \mu_i d\mathfrak{E}_i). \tag{2.24}$$

3. EGB-AdS black holes

3.1 Static solutions

Due to the nonlinearity of the field equations, it is very difficult to find exact solutions of the EGB equations. However, the static spherical and topological black holes are known in closed form, as well as their generalization with an extra U(1) field.

The counterparts of the Schwarzschild solution in EGB theory with negative cosmological constant were found in [36, 37] and have a line element

$$ds^2 = \frac{dr^2}{f(r)} + r^2 d\Sigma_{k,d-2}^2 - f(r) dt^2 \tag{3.1}$$

where the $(d-2)$ -dimensional metric $d\Sigma_{k,d-2}^2$ is

$$d\Sigma_{k,d-2}^2 = \begin{cases} d\Omega_{d-2}^2 & \text{for } k = +1 \\ \sum_{i=1}^{d-2} dx_i^2 & \text{for } k = 0 \\ d\Xi_{d-2}^2 & \text{for } k = -1, \end{cases} \tag{3.2}$$

and $d\Omega_{d-2}^2$ denoting the unit metric on S^{d-2} . By H^{d-2} we will understand the $(d-2)$ -dimensional hyperbolic space, whose unit metric $d\Xi_{d-2}^2$ can be obtained by analytic continuation of that on S^{d-2} .

The function $f(r)$ presents a complicated dependence on ℓ, α (here we restrict to the branch of solutions which are well behaved in the $\alpha \rightarrow 0$ limit)

$$f(r) = k + \frac{2r^2}{\alpha(d-3)(d-4)} \left(1 - \sqrt{1 + \alpha(d-3)(d-4) \left(\frac{\mu}{r^{d-1}} - \frac{1}{\ell^2} \right)} \right), \tag{3.3}$$

with μ a constant.

The Hawking temperature of the black holes is $T_H = f'(r_h)/(4\pi)$, where a prime denotes the derivative with respect the radial coordinate and r_h is the largest positive root of $f(r)$, typically associated to the outer horizon of a black hole.

One can compute the action of these solutions in a simple way by noticing the relation

$$\frac{1}{2} \left(R - 2\Lambda + \frac{1}{4} \alpha L_{\text{GB}}^{(1)} \right) = r^{2-d} \left(-\frac{1}{2} r^{d-2} f' + \frac{1}{4} \alpha (d-2)(d-3) r^{d-4} f'(f-k) \right)'. \tag{3.4}$$

A straightforward computation⁸ shows that the contribution of the bulk action at infinity together with the boundary terms $I_b^{(E)} + I_b^{(GB)} + I_{\text{ct}}^0$ is equal to βM , where M is the mass-energy of the solutions as computed from the boundary stress tensor

$$M = \frac{(d-2)V_{k,d-2}}{16\pi G} \mu + M_k^0. \tag{3.5}$$

⁸The supplementary counterterm I_{ct}^s vanishes for the boundary geometry of the black hole solutions.

In the above expression

$$M_k^0 = \frac{V_{k,d-2}}{16\pi G} \left(\frac{3k^2}{4} \ell_c^2 (3U-2) \delta_{d,5} - \frac{5k}{8} \ell_c^4 \frac{5U-2}{3} \delta_{d,7} + \frac{35k^2}{64} \ell_c^6 \frac{7U-2}{5} \delta_{d,9} + \dots \right) \quad (3.6)$$

is the Casimir mass-energy⁹ and $V_{k,d-2}$ is the (dimensionless) volume associated with the metric $d\Sigma_{k,d-2}^2$. Following [25], one can extrapolate the Casimir term to

$$M_k^0 = \frac{V_{k,d-2}}{8\pi G} (-k)^{(d-1)/2} \frac{(d-2)!!^2}{(d-1)!} \left(\frac{(d-2)U-2}{d-4} \right) \ell_c^{d-3}. \quad (3.7)$$

From (2.23) one finds the EGB black hole entropy

$$S = S_0 + S_c \quad \text{with} \quad S_0 = \frac{A_H}{4G}, \quad S_c = \alpha \frac{V_{k,d-2}}{4G} \frac{k}{2} r_h^{d-4} (d-2)(d-3), \quad (3.8)$$

with the event horizon area $A_H = r_h^{d-2} V_{k,d-2}$. One can easily verify that the first law of thermodynamics (2.24) also holds, with $\mu_i = \mathfrak{E}_i = 0$.

The expressions of mass (without the Casimir term) and entropy agree with previous results in the literature found by using a different approach [39–42]. The Noetherian (or Wald's) approach [43] is particularly interesting, presenting an expression of S which is holographic in spirit [40]. As discussed in [41], the entropy of the black hole solutions (3.1) can be written as a integral over the event horizon

$$S = \frac{1}{4G} \int_{\Sigma_h} d^{d-2} x \sqrt{\tilde{h}} \left(1 + \frac{\alpha}{2} \tilde{R} \right), \quad (3.9)$$

(where \tilde{h} is the determinant of the induced metric on the horizon and \tilde{R} is the event horizon curvature), which agrees with the result (3.8). This relation appears to be universal, being satisfied by the other EGB black objects discussed in this work.

The stress tensor of the dual theory computed according to (2.22) has the same expression as in the $\alpha = 0$ case

$$8\pi G \langle \tau_a^b \rangle = \frac{M}{2\ell_c^{d-2}} (\delta_a^b + (d-1) u_a u^b), \quad (3.10)$$

where $u_a = \delta_a^t$. This tensor is finite, covariantly conserved and manifestly traceless and presents a nontrivial dependence of the GB coefficient α .

The GB term gives rise to some interesting effects on the thermodynamics of black holes in AdS space. First, as observed in [15, 38], for $k = -1$ topological black holes, the second term in (3.8) can make the whole expression negative for sufficiently small black holes. In the $k = 0$ case, where the horizon is flat, the GB term has no effect on the expression for entropy, which is simply the area of the horizon. For spherically symmetric solutions, a locally stable small black holes branch appears for $d = 5$, which is absent in the case without the GB term. However, for $d \geq 6$, the thermodynamic behavior of the EGB black holes is qualitatively similar to the case with $\alpha = 0$. Detailed discussions of the thermodynamics of EGB black holes can be found in [15, 16, 39].

⁹Note that the expression of the Casimir energy agrees with that found in [11] by using Kounterterms regularisation.

3.2 Rotating black holes with equal magnitude angular momenta

The computation of the global charges and entropy of the rotating black holes in EGB theory represents another nontrivial application of the general formalism in section 2. No exact solutions are available in this case,¹⁰ except for the $k = 0$ configurations in [45]. However, these exact solutions are essentially obtained by boosting the static configurations with flat horizon; thus locally they are equivalently to the static ones but not globally.

In this subsection we consider a class of EGB black holes in a odd number of spacetime dimensions $d = 2N + 1$ ($N \geq 2$), possessing equal magnitude angular momenta and a spherical horizon topology. This factorizes the angular dependence [46] and reduces the problem to studying the solutions of four differential equations with dependence only on the radial variable r . The explicit computation of the black holes' action and boundary stress tensor also simplifies drastically in this case.

These solutions are found for the following parametrization of the metric¹¹

$$\begin{aligned}
 ds^2 = & -b(r)dt^2 + \frac{dr^2}{f(r)} + g(r) \sum_{i=1}^{N-1} \left(\prod_{j=0}^{i-1} \cos^2 \theta_j \right) d\theta_i^2 \\
 & + h(r) \sum_{k=1}^N \left(\prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k (d\varphi_k - w(r)dt)^2 \\
 & + (g(r) - h(r)) \left\{ \sum_{k=1}^N \left(\prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k d\varphi_k^2 - \left[\sum_{k=1}^N \left(\prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k d\varphi_k \right]^2 \right\},
 \end{aligned} \tag{3.11}$$

where $\theta_0 \equiv 0$, $\theta_i \in [0, \pi/2]$ for $i = 1, \dots, N-1$, $\theta_N \equiv \pi/2$, $\varphi_k \in [0, 2\pi]$ for $k = 1, \dots, N$. The metric gauge choice we consider here is $h(r) = r^2$. Note also that the $k = 1$ static black holes (3.1) are recovered for $w(r) = 0$, $h(r) = g(r) = r^2$ and $b = f$ as given by (3.3).

Although only the $d = 5$ case has been studied so far in the literature by using numerical methods [17], similar solutions should exist for all N . A discussion of the properties of these stationary black holes is beyond the purposes of this work. Here we shall present only a computation of their mass-energy, angular momentum and entropy, in which case only the asymptotic form of the metric is required, looking for the effects at this level of the GB term.¹²

By solving the EGB field equations (2.3) for large r , one find that these rotating black

¹⁰Slowly rotating EGB solutions have been considered recently in [44] within a perturbative approach.

¹¹For $\alpha = 0$ (*i.e.* no GB term), these solutions are a subset of the general configurations with the maximal number of rotation parameters discussed in [47], with $f(r) = 1 + r^2/\ell^2 - 2\hat{M}\Xi/r^{d-3} + 2\hat{M}\hat{a}^2/r^{d-1}$, $h(r) = r^2 \left(1 + 2\hat{M}\hat{a}^2/r^{d-1} \right)$, $w(r) = 2\hat{M}\hat{a}/r^{d-3}h(r)$, $g(r) = r^2$, $b(r) = r^2 f(r)/h(r)$, where \hat{M} and \hat{a} are two constants related to the solution's mass and angular momentum and $\Xi = 1 - \hat{a}^2/\ell^2$.

¹²For a different computation of mass and angular momentum of rotating black holes in EGB gravity, see [48, 49].

holes present the following asymptotics in terms of the arbitrary constants $\bar{f}, \bar{b}, \bar{j}$

$$f = 1 + \frac{r^2}{\ell_c^2} + \bar{f} \left(\frac{\ell_c}{r}\right)^{d-3} + \dots, \quad b = 1 + \frac{r^2}{\ell_c^2} + \bar{b} \left(\frac{\ell_c}{r}\right)^{d-3} + \dots, \quad (3.12)$$

$$h/r^2 = 1 + (\bar{f} - \bar{b}) \left(\frac{\ell_c}{r}\right)^{d-3} + \dots, \quad w(r) = \frac{\bar{j}}{r} \left(\frac{\ell_c}{r}\right)^{d-2} + \dots,$$

while the near horizon expansion of the nonextremal solutions is

$$f(r) = f_1(r - r_h) + O(r - r_h)^2, \quad h(r) = h_h + O(r - r_h), \quad (3.13)$$

$$b(r) = b_1(r - r_h) + O(r - r_h)^2, \quad w(r) = w_h + O(r - r_h),$$

with f_1, b_1, h_h positive constants. For the solutions within the ansatz (3.11), the event horizon's angular velocities are all equal, $\Omega_k = \Omega_H = w(r)|_{r=r_h}$. The Killing vector $\chi = \partial/\partial t + \sum_k \Omega_k \partial/\partial \varphi_k$ is orthogonal to and null on the horizon.

The Hawking temperature and the event horizon area of these configurations are

$$T_H = \frac{\sqrt{b'(r_h)f'(r_h)}}{4\pi}, \quad A_H = r_h^{d-3} \sqrt{h_h} V_{1,d-2}. \quad (3.14)$$

The conserved charges of the rotating black holes are obtained by using again the counterterm method in conjunction with the quasilocal formalism. Based on the results we have obtained for spacetime dimensions $d = 5, 7, 9$, we propose the following general expression for mass-energy and angular momentum:¹³

$$M = \frac{\ell_c^{d-3} U}{16\pi G} [\bar{f} - (d-1)\bar{b}] V_{1,d-2} + M_1^0, \quad J_{(k)} = J = \frac{\ell_c^{d-2}}{8\pi G_d} \bar{j} U V_{1,d-2}, \quad (3.15)$$

with M_1^0 the Casimir energy as given by (3.7). The entropy of these solutions as computed from the general relation (2.23) (with $\mu_i = J$, $\mathfrak{E}_i = \Omega_H$) is

$$S = S_0 + S_c, \quad \text{with } S_0 = \frac{A_H}{4G}, \quad S_c = \alpha \frac{V_{1,d-2}}{8G} (d-3) \sqrt{h_h} r_h^{d-5} \left(d - 1 - \frac{h_h}{r_h^2}\right). \quad (3.16)$$

It is interesting to note that the entropy (3.16) can also be written in the Wald's form (3.9). However, the derivation of (3.9) in ref. [41] covers the case of static solutions only; it would be interesting to extend it by including the effects of rotation. Of course, the ultimate test of the formulae (3.15), (3.16) will be possible when the exact solutions will be found (*i.e.* the relation between b_1, f_1, h_h and $\bar{f}, \bar{b}, \bar{j}$) and verify that, with these definitions, the first law of thermodynamics is satisfied.

The boundary metric upon which the dual field theory resides corresponds to a static Einstein universe in $(d-1)$ dimensions with line element $h_{ab} dx^a dx^b = \ell_c^2 d\Sigma_{1,d-2}^2 - dt^2$. The stress tensor for the boundary dual theory has also an interesting form. Restricting to $d = 5$,

¹³Note that these quantities are evaluated in a frame which is non-rotating at infinity.

one finds (with $x^1 = \theta_1$, $x^2 = \varphi_1$, $x^3 = \varphi_2$, $x^4 = t$ and $d\Sigma_{1,3}^2 = d\theta_1^2 + \sin^2 \theta_1 \varphi_1^2 + \cos^2 \theta_1 \varphi_2^2$)

$$\begin{aligned}
 8\pi G \langle \tau_b^a \rangle = & \left(\frac{3U - 2}{8\ell_c} - \frac{\bar{f}U}{2\ell_c} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \frac{2U}{\ell_c} (\bar{b} - \bar{f}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2 \theta_1 & \cos^2 \theta_1 & 0 \\ 0 & \sin^2 \theta_1 & \cos^2 \theta_1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
 & + 2\hat{j}U \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^2 \theta_1 \\ 0 & 0 & 0 & \cos^2 \theta_1 \\ 0 & -\frac{1}{\ell_c^2} & -\frac{1}{\ell_c^2} & 0 \end{pmatrix}. \tag{3.17}
 \end{aligned}$$

As expected, this traceless stress tensor is finite and covariantly conserved.

4. EGB-AdS black string solutions

The $k = 1$ Schwarzschild-AdS(-GB) black hole solutions in d -dimensions have an event horizon of topology S^{d-2} , which matches the S^{d-2} topology of the spacelike infinity. Horowitz and Copsey found in [19] a different type of $d = 5$ solution of Einstein equations with negative Λ , with an event horizon topology $S^2 \times S^1$. The configurations there have no dependence on the ‘compact’ extra dimension, and their conformal boundary is the product of time and $S^2 \times S^1$. These solutions have been generalized to higher dimensions $d \geq 5$ in [20], configurations with an event horizon topology $H^{d-3} \times S^1$ being considered as well.

Black objects with event horizon topology $S^{d-3} \times S^1$ matching that of the spacelike infinity are familiar from the $\Lambda = 0$ physics and they are usually called black strings [50]. The solutions in [19, 20] present many similar properties with the $\Lambda = 0$ case, and are naturally interpreted as the AdS counterparts of these configurations.¹⁴ Although the AdS black strings are not known in closed form,¹⁵ one can analyse their properties by using a combination of analytical and numerical methods, which is enough for most purposes.

Different from the $\Lambda = 0$ limit, it was found in [20] that the AdS black string solutions with an event horizon topology $S^{d-3} \times S^1$ have a nontrivial, vortex-like globally regular limit with zero event horizon radius. As argued in [19, 20], these solutions provide the gravity dual of a field theory on a $S^{d-3} \times S^1 \times S^1$ (or $H^{d-3} \times S^1 \times S^1$) background.

Generalizations of AdS black strings with gauge fields are studied in [53–56]. The ref. [54] discussed also the properties of a set of rotating solutions. The issue of Gregory-Laflamme instability [57] for AdS black strings in Einstein gravity was addressed in [58], nonuniform solutions (*i.e.* with dependence on the compact ‘extra’-dimension) being constructed at the perturbative level in [59].

In this section we present arguments for the existence of asymptotically AdS black strings in EGB theory and analyse their basic properties. This provides also an interesting

¹⁴Note that these configurations are very different from the warped AdS solutions as discussed for instance in [51], although the latter are also usually called black strings in the literature. General remarks about the properties of these solutions in Lovelock gravity are presented in [52].

¹⁵See, however, the $d = 5$ supersymmetric Einstein-U(1) magnetic black string exact solutions in [53].

application of the formalism in section 2, since the AdS black strings have no obvious background. For $d = 5$, the work here generalizes for $\Lambda < 0$ the Kaluza-Klein EGB black strings discussed in ref. [60].

4.1 The metric ansatz and reduced action

We consider the following parametrization of the d -dimensional line element (with $d \geq 5$)

$$ds^2 = a(r)dz^2 + \frac{dr^2}{f(r)} + g^2(r)d\Sigma_{k,d-3}^2 - b(r)dt^2, \quad (4.1)$$

the 'extra'-direction z being compact¹⁶ with period L . For $g(r) = r$, this reduces to the metric ansatz in [20].

Without fixing a metric gauge, a straightforward computation leads to the following reduced action of the system

$$A_{\text{eff}} = \int dr dt L_{\text{eff}}, \quad \text{with} \quad L_{\text{eff}} = L_E + \frac{\alpha}{4}L_{\text{GB}}, \quad (4.2)$$

where

$$\begin{aligned} L_E = & (d-3)(d-4)\sqrt{\frac{ab}{f}}g^{d-5}(k + fg'^2) + \frac{1}{2}\sqrt{\frac{f}{ab}}g^{d-3}a'b' \\ & + (d-3)\sqrt{abf}g^{d-4}\left(\frac{a'}{a} + \frac{b'}{b}\right)g' - 2\Lambda g^{d-3}\sqrt{\frac{ab}{f}}, \end{aligned} \quad (4.3)$$

$$\begin{aligned} L_{\text{GB}} = & (d-3)(d-4)\sqrt{\frac{f}{ab}}g^{d-5}a'b'(k - fg'^2) \\ & - \frac{2}{3}(d-3)(d-4)(d-5)\sqrt{abf}g^{d-6}\left(\frac{a'}{a} + \frac{b'}{b}\right)g'(fg'^2 - 3k) \\ & + \frac{1}{3}(d-3)(d-4)(d-5)(d-6)\sqrt{\frac{ab}{f}}g^{d-7}(3k^2 + 6kfg'^2 - f^2g'^4). \end{aligned} \quad (4.4)$$

The corresponding equations for the metric functions a, b, f are found by taking the variation of A_{eff} with respect to a, b, f and g and fixing after that the metric gauge $g(r) = r$ (this is equivalent to directly solving the EGB equations, but technically simpler). The resulting relations are very long and we do not include them here. Similar to the case of Einstein gravity, the equations for a and f are first order while the equation for b is second order.

4.2 The asymptotics

We consider non-extremal black string solutions presenting the following expansion near the event horizon (taken at constant $r = r_h$)

$$a(r) = a_h + O(r - r_h), \quad b(r) = b_1(r - r_h) + O(r - r_h)^2, \quad f(r) = f_1(r - r_h) + O(r - r_h)^2, \quad (4.5)$$

¹⁶For these uniform black string solutions, the period L is an arbitrary positive constant and plays no role in our results. However, similar to the $\Lambda = 0$, its value is crucial when discussing the issue of Gregory-Laflamme instability [58, 59] of these objects.

with all coefficients fixed by the parameters a_h, b_1 . The condition for a regular event horizon is $f'(r_h) > 0, b'(r_h) > 0$ and a_h a positive constant. In the $k = -1$ case, this implies the existence of a minimal value of the event horizon radius.

We consider solutions of the equations of motion whose boundary topology is the product of time and $S^{d-3} \times S^1, R^{d-3} \times S^1$ or $H^{d-3} \times S^1$. For even values of the spacetime dimension d , the solutions of the field equations admit at large r a power series expansion of the form:

$$\begin{aligned}
 a(r) &= \left(\frac{d-4}{d-3}\right)k + \frac{r^2}{\ell_c^2} + \sum_{j=1}^{(d-4)/2} a_j \left(\frac{\ell_c}{r}\right)^{2j} + c_z \left(\frac{\ell_c}{r}\right)^{d-3} + O(1/r^{d-2}), \\
 b(r) &= \left(\frac{d-4}{d-3}\right)k + \frac{r^2}{\ell_c^2} + \sum_{j=1}^{(d-4)/2} a_j \left(\frac{\ell_c}{r}\right)^{2j} + c_t \left(\frac{\ell_c}{r}\right)^{d-3} + O(1/r^{d-2}), \\
 f(r) &= \frac{k(d-1)(d-4)}{(d-2)(d-3)} + \frac{r^2}{\ell_c^2} + \sum_{j=1}^{(d-4)/2} f_j \left(\frac{\ell_c}{r}\right)^{2j} + (c_z + c_t) \left(\frac{\ell_c}{r}\right)^{d-3} + O(1/r^{d-2}),
 \end{aligned} \tag{4.6}$$

where a_j, f_j are constants depending on the index k and the spacetime dimension only. Specifically, we find

$$a_1 = \frac{k^2(4U-1)}{27U}, \quad f_1 = \frac{(59U-11)k^2}{216U} \quad \text{for } d=6, \tag{4.7}$$

$$a_1 = \frac{k^2(49U-9)}{1125U}, \quad f_1 = \frac{(177U-17)k^2}{2250U}, \tag{4.8}$$

$$a_2 = -\frac{2k(63-U(1381-8118U))}{253125U^2}, \quad f_2 = -\frac{8k(9-U(178-1269U))}{84375U^2}, \quad \text{for } d=8,$$

their expression becoming more complicated for higher d , with no general pattern becoming apparent.

Similar to the $\alpha = 0$ case, the corresponding expansion for odd values of d contains logarithmic terms

$$\begin{aligned}
 a(r) &= \left(\frac{d-4}{d-3}\right)k + \frac{r^2}{\ell_c^2} + \sum_{j=1}^{(d-5)/2} \bar{a}_j \left(\frac{\ell_c}{r}\right)^{2j} + \zeta \log\left(\frac{r}{\ell_c}\right) \left(\frac{\ell_c}{r}\right)^{d-3} \\
 &\quad + c_z \left(\frac{\ell_c}{r}\right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right),
 \end{aligned} \tag{4.9}$$

$$\begin{aligned}
 b(r) &= \left(\frac{d-4}{d-3}\right)k + \frac{r^2}{\ell_c^2} + \sum_{j=1}^{(d-5)/2} \bar{a}_j \left(\frac{\ell_c}{r}\right)^{2j} + \zeta \log\left(\frac{r}{\ell_c}\right) \left(\frac{\ell_c}{r}\right)^{d-3} \\
 &\quad + c_t \left(\frac{\ell_c}{r}\right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right),
 \end{aligned} \tag{4.10}$$

$$\begin{aligned}
 f(r) &= \frac{k(d-1)(d-4)}{(d-2)(d-3)} + \frac{r^2}{\ell_c^2} + \sum_{j=1}^{(d-5)/2} \bar{f}_j \left(\frac{\ell_c}{r}\right)^{2j} + 2\zeta \log\left(\frac{r}{\ell_c}\right) \left(\frac{\ell_c}{r}\right)^{d-3} \\
 &\quad + (c_z + c_t + c_0) \left(\frac{\ell_c}{r}\right)^{d-3} + O\left(\frac{\log r}{r^{d-1}}\right),
 \end{aligned} \tag{4.11}$$

where, restricting to the first two values of d

$$c_0 = \frac{k^2(1-U)}{36U} \quad \text{for } d=5, \quad c_0 = \frac{k^3(35U-17)}{3200U} \quad \text{for } d=7, \quad (4.12)$$

and

$$\zeta = \frac{k^2}{12} \quad \text{for } d=5, \quad \zeta = \frac{k(3-57U)}{1600U} \quad \text{for } d=7, \quad (4.13)$$

$$\bar{a}_1 = \frac{k^2(23U-5)}{320U}, \quad \bar{f}_1 = \frac{k^2(52U-7)}{400U} \quad \text{for } d=7. \quad (4.14)$$

For any value of d , the large r form of the solutions presents two arbitrary parameters c_t and c_z which uniquely fix the coefficients of all terms decaying faster than $1/r^{d-3}$.

4.3 The conserved charges and entropy

Apart from the mass-energy M , these solutions possess a second charge associated with the compact z direction, corresponding to the black string's tension \mathcal{T} . The computation of the boundary stress tensor T_{ab} based on the relations in section 2 is straightforward and we find the following expressions for mass and tension (the relations here extrapolate the results found for $5 \leq d \leq 9$)

$$M = M_0 + M_c^{(k,d)}, \quad M_0 = \frac{\ell_c^{d-4}}{16\pi G} [c_z - (d-2)c_t] ULV_{k,d-3}, \quad (4.15)$$

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_c^{(k,d)}, \quad \mathcal{T}_0 = \frac{\ell_c^{d-4}}{16\pi G} [(d-2)c_z - c_t] UV_{k,d-3}, \quad (4.16)$$

where $M_c^{(k,d)}$ and $\mathcal{T}_c^{(k,d)}$ are Casimir-like terms which appear for an odd spacetime dimension only,

$$M_c^{(k,d)} = -L\mathcal{T}_c^{(k,d)} = \frac{\ell_c^{d-4}}{8\pi G} V_{k,d-3} L \left(\frac{3-2U}{24} k^2 \delta_{d,5} - \frac{10+U(123U-7)}{3200} k \delta_{d,7} + \frac{U(75920+U(-284038+2262727U))-1484}{64012032U^2} k^2 \delta_{d,9} + \dots \right). \quad (4.17)$$

The Hawking temperature and the event horizon area of the black strings are

$$T_H = \frac{\sqrt{b'(r_h)f'(r_h)}}{4\pi}, \quad A_H = r_h^{d-3} V_{k,d-3} L \sqrt{a_h}. \quad (4.18)$$

To evaluate the black string's action, it is important to use the observation that one can write

$$R_t^t + \frac{\alpha}{4} \left(H_t^t + \frac{1}{2} L_{\text{GB}} \right) = \frac{1}{r^{d-2}} \sqrt{\frac{f}{ab}} \frac{d}{dr} \left(-\frac{1}{2} r^{d-5} b' \sqrt{\frac{af}{b}} \times \left(r^2 - \frac{1}{2} \alpha (d-3) \left((d-4)(f-k) + f \frac{ra'}{a} \right) \right), \quad (4.19)$$

$$R_z^z + \frac{\alpha}{4} \left(H_z^z + \frac{1}{2} L_{\text{GB}} \right) = \frac{1}{r^{d-2}} \sqrt{\frac{f}{ab}} \frac{d}{dr} \left(-\frac{1}{2} r^{d-5} a' \sqrt{\frac{bf}{a}} \times \left(r^2 - \frac{1}{2} \alpha (d-3) \left((d-4)(f-k) + f \frac{rb'}{b} \right) \right). \quad (4.20)$$

Integrating the first relation above taken together with the field equations (2.3), we isolate the bulk action contribution at infinity and at $r = r_h$. The divergent contributions given by the surface integral term at infinity are canceled by $I_b^{(E)} + I_b^{(GB)} + I_{ct}$ (with $I_{ct} = I_{ct}^0 + I_{ct}^s$), which results in a finite expression of I_{cl} . Together with (2.23), we find the entropy of solutions

$$S = S_0 + S_c, \quad \text{with} \quad S_0 = \frac{A_H}{4G}, \quad S_c = \frac{k}{4G} \frac{\alpha}{2} (d-3)(d-4)r_h^{d-5} LV_{k,d-3} \sqrt{a_h}. \quad (4.21)$$

Therefore, similar to the black hole case, the entropy of a black string is not simply proportional to the black hole horizon area as it is in the Einstein gravity, but has an extra term proportional to the GB coupling parameter (note that, as expected, the GB contribution to S is proportional to the extra-term in the entropy of a $(d-1)$ -dimensional Schwarzschild-GB-AdS black hole (3.1)). Moreover, the entropy of a black string can also be formally written in Wald's form (3.9), the GB contribution being proportional again with the Ricci scalar of the event horizon.

By using the second relation in (4.19) we find also that $I = -\beta T L$. This relation together with (2.23) leads to an unexpectedly simple Smarr-type formula, relating quantities defined at infinity to quantities defined at the event horizon:

$$M + T L = T_H S, \quad (4.22)$$

which is the result found in [20] for solutions without a GB term. This relation also provided a useful check of the accuracy of the numerical solutions we have obtain.

The first law of thermodynamics (2.24) looks more complicated for black strings as compared to the black hole case, since the length scale L enters there as an extensive parameter [62], *i.e.* $\mu = T$, $\mathfrak{C} = L$. In the absence of closed form solutions, the validity of the generic relation (2.24) can be verified only numerically. In practice, for both $d = 5$ and $d = 6$ solutions, we have integrated the first law for a fixed period L and computed a value of mass. Then the expression of the tension is computed¹⁷ from the Smarr-type formula (4.22). The values of M and T computed in this way were found to coincide with a reasonable accuracy with those derived by using the expression (4.15), up to the overall Casimir terms M_0, T_0 .

We give here also the expectation value of the stress tensor of the dual theory for the simplest case $d = 5$ (with the background metric upon which the dual field theory resides $h_{ab} dx^a dx^b = \ell_c^2 d\Sigma_{k,2}^2 + dz^2 - dt^2$, and $x^1 = \theta, x^2 = \phi, x^3 = z, x^4 = t$, while θ, ϕ are the coordinates on a surface of constant r, t)

$$8\pi G \langle \tau_b^a \rangle = k^2 \left(\begin{array}{cccc} \frac{3-U}{24\ell_c} & 0 & 0 & 0 \\ 0 & \frac{3-U}{24\ell_c} & 0 & 0 \\ 0 & 0 & \frac{2U-3}{24\ell_c} & 0 \\ 0 & 0 & 0 & \frac{2U-3}{24\ell_c} \end{array} \right) + \left(\begin{array}{cccc} -\frac{(c_t+c_z)U}{2\ell_c} & 0 & 0 & 0 \\ 0 & -\frac{(c_t+c_z)U}{2\ell_c} & 0 & 0 \\ 0 & 0 & \frac{(3c_z-c_t)U}{2\ell_c} & 0 \\ 0 & 0 & 0 & \frac{(3c_t-c_z)U}{2\ell_c} \end{array} \right).$$

¹⁷This is the standard approach used to compute the mass and tension of the $\Lambda = 0$ black string solutions presenting a dependence on the extra-coordinate z , in which case it has proven very difficult to read M and T from the asymptotic data at infinity, see *e.g.* [63].

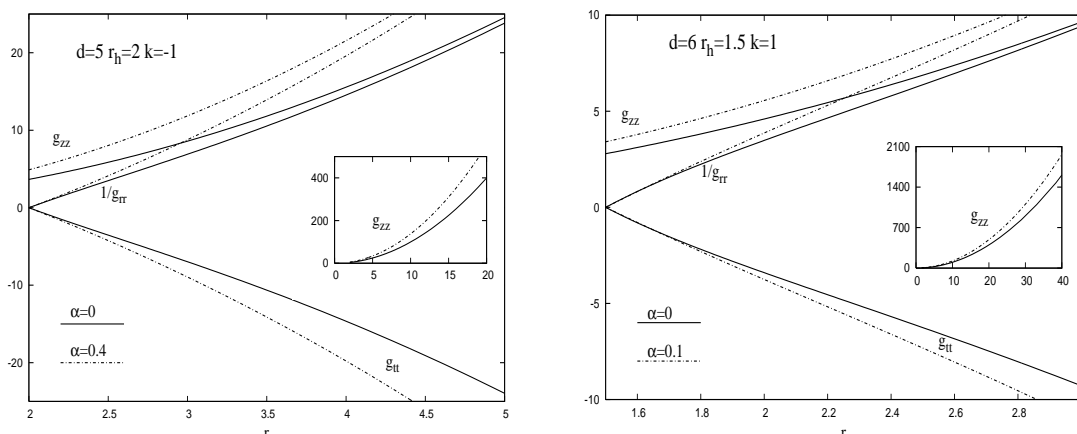


Figure 1: The profiles of the metric functions g_{tt} , g_{zz} and $1/g_{rr}$ are shown for typical $k = \pm 1$ black string solutions in EGB theory. For comparison, we included also the profiles of the corresponding solutions in Einstein gravity.

As expected, this anisotropic perfect fluid stress tensor is finite and covariantly conserved. However, for $k \neq 0$ it is *not* traceless, with

$$8\pi G \langle \tau_a^a \rangle = \frac{\ell_c^3}{8} \left(\left(1 - \frac{4U}{3} \right) R^2 + (5U - 4) R_{ab} R^{ab} + (1 - U) R_{abcd} R^{abcd} \right), \quad (4.23)$$

(where the geometric quantities are computed for the metric h_{ab}). For $U = 1$ (*i.e.* $\alpha = 0$), this trace is equal to the conformal anomaly of the boundary CFT [61]. A similar computation performed for the $d = 7, 9$ cases leads again to a nonvanishing trace of the boundary stress tensor. As discussed at length in [54], the trace of $\langle \tau_a^a \rangle$ for the seven-dimensional Einstein gravity black string solutions matches precisely the conformal anomaly of the dual six-dimensional superconformal $(2, 0)$ theory [61, 64].

One can see that the coupling constant α enters the expression of the trace anomaly and Casimir energy for both EGB black holes and black strings. This is what we expect on general grounds,¹⁸ since in EGB theory the effective radius (2.8) of the AdS space and the boundary metric depend on α .

We close this part by remarking that by performing the double analytic continuation $z \rightarrow iu$, $t \rightarrow i\chi$ with a simultaneous exchange of corresponding charges, the black strings become static bubble of nothing solutions in EGB theory. In order to obtain a regular solution, the spatial coordinate χ is identified with a period $\beta = 1/T_H$.

4.4 Numerical results

In the absence of closed form solutions, we relied in this case on numerical methods to solve the EGB equations. The numerical methods here are similar to those used in literature to study other $\Lambda < 0$ black string solutions [54, 58, 59]. Taking units such that $G = 1$, we used

¹⁸However, a more realistic string theory computation would require to consider solutions in a different model than (2.1), containing α^3 corrections consisting in Weyl-squared terms.

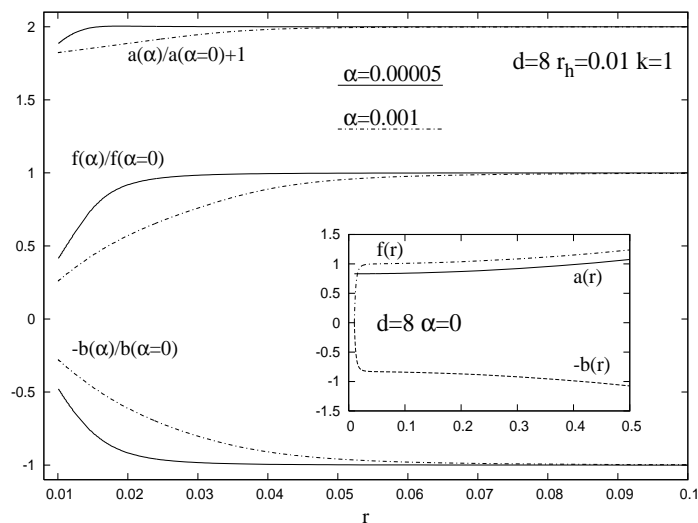


Figure 2: The effects on the solutions of a nonzero Gauss-Bonnet coupling constant is plotted for two $k = 1$ black strings in $d = 8$. The small r region of the profiles of the metric function in Einstein gravity is also plotted.

a standard solver which involves a Newton-Raphson method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure [65]. Typical mesh sizes include $10^3 - 10^4$ points. The solutions have a typical relative accuracy of 10^{-8} .

For $k = 0$, the EGB equations admit the exact solution $a = r^2$, $b = f$ (where f is given by (3.3)), which was recovered by our numerical procedure. This solution is likely to be unique, corresponding to the known planar topological black hole (4.1). Therefore we shall concentrate here on the cases $k = \pm 1$.

In order to construct numerical solutions for a given d , the constants (α, Λ) have to be fixed. Then the solution is further specified by the event horizon r_h . Given the complexity of the problem, the complete classification of the solutions in the space of parameters is a considerable task that is not aimed in this paper. Instead, we set $\ell = 1$ by using a suitable rescaling, and analyse in detail a few particular classes of solutions, which hopefully would reflect all relevant properties of the general pattern. Also, we shall restrict in this work to the study of the branch of solutions smoothly emerging from the Einstein gravity configurations. Although most of the numerical data presented here corresponds to $d = 5, 6$, in which cases new qualitative features exist, we have found also solutions for $d = 7, 8$. Therefore we conjecture that they exist for any $d \geq 5$.

For all configurations we have studied, the metric functions $a(r)$, $b(r)$ and $f(r)$ interpolate monotonically between the corresponding values at $r = r_h$ and the asymptotic values at infinity, without presenting any local extrema. The profiles of the metric functions of typical EGB black string solutions are presented on figures 1,2 together with the corresponding data in Einstein gravity. One can see that a nonzero α leads to a deformation of all metric functions at all scales and is particularly apparent on the function $g_{zz} = a(r)$.

When studying the dependence of the black strings on the GB parameter α and the event horizon radius r_h we have found that they present an unexpectedly rich structure.

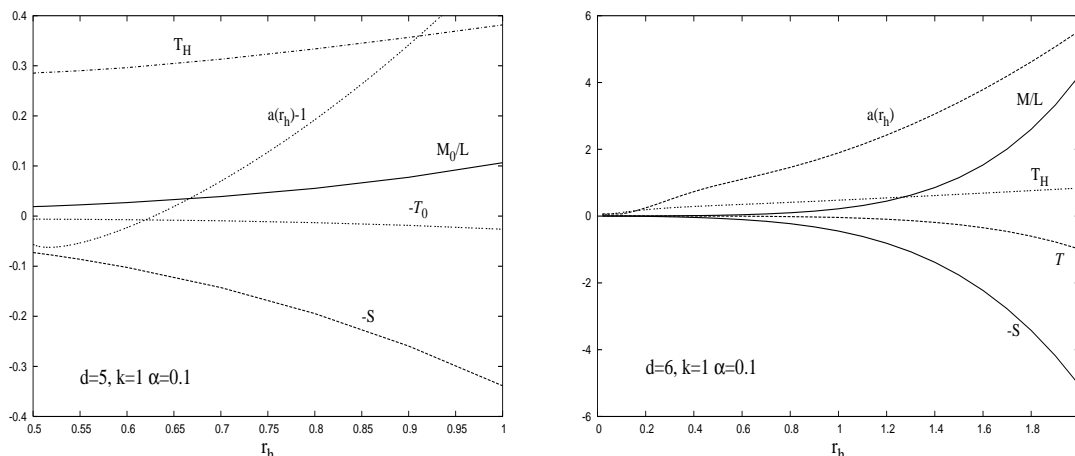


Figure 3: The mass-parameter M , the tension \mathcal{T} (without the Casimir terms), the value of the metric function $a(r)$ at the event horizon as well as the Hawking temperature T_H and the entropy S of $k = 1$ black string solutions with $\alpha = 0.1$ are represented as functions of the event horizon radius in $d = 5, 6$ dimensions.

As discussed in [20, 54], the AdS black strings in d -dimensions follow the general pattern of the black holes with the same index k in $(d - 1)$ dimensions. This remains valid for EGB configurations. Therefore the cases $d = 5, 6$ with $k = 1$ are rather special, since there are no EGB black holes in four dimensions, while the $d = 5$ EGB black holes with spherical horizon have distinct properties.

Before discussing the pattern of solutions, let us briefly recall the structure of the $k = 1$ black strings in the Einstein theory. In [20] it was shown that a family of black strings exist presenting a regular horizon at $r = r_h$ for all $r_h > 0$. On the other hand, the equations also admit a vortex-like regular solution on $r \in [0, \infty)$ with $f(0) = 1$, $a(0) > 0$, $b(0) > 0$. In the limit $r_h \rightarrow 0$, the black strings approach this regular solution. In particular, the derivatives $f'(r_h)$ and $b'(r_h)$ both diverge for $r_h \rightarrow 0$. However, the regular solution has a finite and nonzero mass and tension, for any value of d .

This picture is very different in the EGB theory. First, our numerical results show that the $d = 5, 6$ $k = 1$ globally regular solutions with $r_h = 0$, which exist for $\alpha = 0$, do not appear in the EGB theory. For $d = 5$ this can be understood by noticing the existence of a minimal allowed value of r_h , which results from the expression of the parameter f_1 in the event horizon expansion (4.5)

$$f_1 = \frac{r_h^2(\ell^2 + 2\alpha)}{\ell^2\alpha} - \frac{1}{\ell^2\alpha} \sqrt{(\ell^2 - 2\alpha)(r_h^2(\ell^2 - 2\alpha) - 2\ell^2\alpha)} > 0, \tag{4.24}$$

(the general d -expression is much more complicated), which implies $r_h > \ell/\sqrt{\ell^2/(2\alpha) - 1}$. Thus there exist a minimal value of r_h for which the Einstein black strings can be generalized into EGB ones.¹⁹ This is a nonperturbative effect, which cannot be seen consider-

¹⁹As discussed in [60], the $d = 5$ black strings in Kaluza-Klein theory ($\Lambda = 0$) present also a minimal horizon size $r_{h(min)} = \sqrt{2\alpha}$. This can be seen by taking the $\ell \rightarrow \infty$ limit in the AdS relation (4.24).

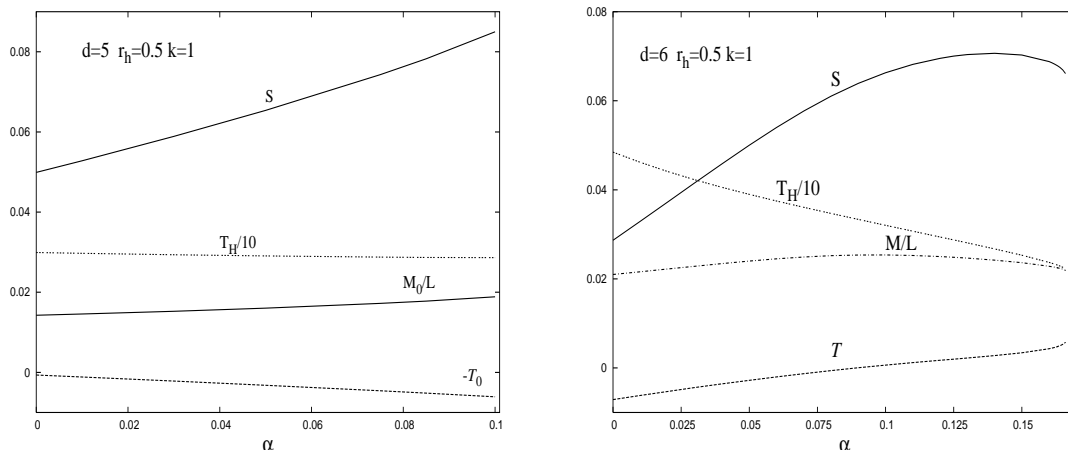


Figure 4: The Hawking temperature T_H , the mass-energy M , the tension \mathcal{T} and the entropy S are plotted as a function of α for $d = 5, 6$ black string with $k = 1$.

ing the GB term as a small deformation of the Einstein gravity, in which case one finds $f_1 = 1/r_h + 4r_h/\ell^2 + \alpha/(2r_h^3) + O(\alpha)^2$.

The pattern of $k = 1$ solutions is different for $d = 6$. Fixing $r_h > 0$, we could construct for all $r_h > 0$ a family of solution for $\alpha \in [0, \alpha_{\max}]$. In the limit $\alpha \rightarrow 0$, the Einstein theory solution is approached smoothly (different from the $d = 5$ case discussed above). The limit $r_h \rightarrow 0$ is different, however, from the $\alpha = 0$ case, since no globally regular solution is found in this case. Instead, $a(0) \rightarrow 0$ while $f'(r_h)$ and $b'(r_h)$ take finite (and very small) values in this limit. As a result, the Ricci scalar diverges and a naked singular configuration is approached.²⁰ These different observations clearly suggest that the two limits $\alpha \rightarrow 0$ and $r_h \rightarrow 0$ do not intercommute in the pattern of solutions available for $d = 6$.

Of course, the results above do not prove the absence of $k = 1$ globally regular configurations in EGB theory with negative Λ (although unlikely, they may be disconnected from the studied black string branch). For example, for any d it is possible to write a consistent expansion of the solution near the origin as a power series in r . However, for both $d = 5$ and $d = 6$ we have failed to find such solutions, when considering the corresponding boundary value problem.

The situation appears to be different for $d > 6$ (and $k = 1$), in which case the $\alpha = 0$ pattern found in [20] is still valid. There the black string event horizon radius is an arbitrary parameter. Our numerics suggest that, for a given $\alpha \leq \alpha_{\max}$, a globally regular solution is approached as $r_h \rightarrow 0$, while the parameters $f'(r_h)$, $b'(r_h)$ diverge in this limit (therefore also the Hawking temperature), while $a(r_h)$ remains finite and nonzero. These features are presented in figure 2, where we plot the data for two $d = 8$ black string with a small event horizon radius $r_h = 0.01$ together with the corresponding solution in Einstein gravity²¹ (the

²⁰For both $d = 5$ and $d = 6$, we have found rather difficult to approach the limiting r_h configurations by employing the ansatz (4.1). A different metric parametrization appears to be necessary.

²¹One can see that, as implied by (4.6), given the small values of α for the two profiles in figure 2, the ratio between the metric functions in EGB theory and those in Einstein gravity is very close to one for large enough values of the radial coordinate.

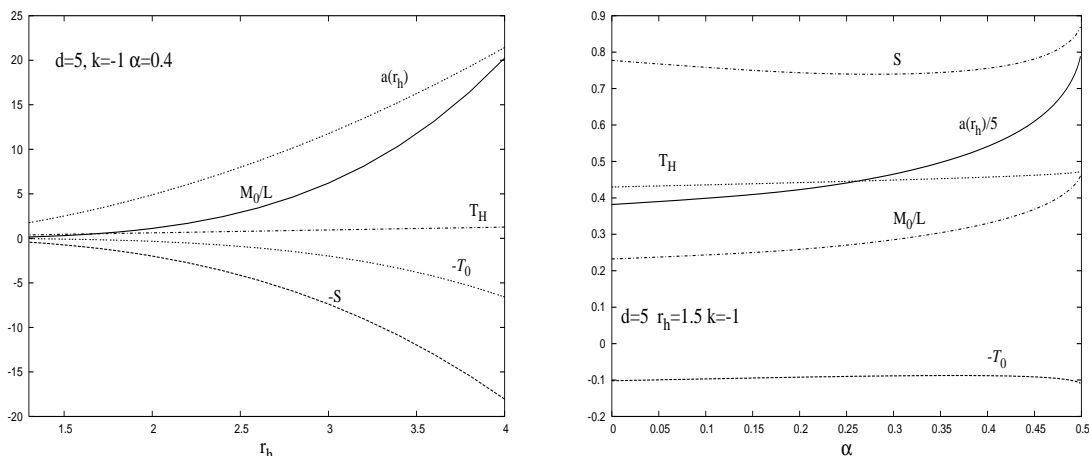


Figure 5: The dependence of the solutions' parameters on the event horizon radius r_h and Gauss-Bonnet parameter α is presented for $k = -1$ topological black strings in five dimensions.

profiles of the metric functions for large r_h look similar to those in figure 1 (with $k = 1$). One can see that both $f(r)$ and $b(r)$ present a sharp transition in a small region near the horizon, where $a(r)$ remain almost constant and nonzero. However, the severe numerical difficulties encountered for small r_h solutions with $\alpha \neq 0$ prevented us from analyzing in details the black string-globally regular vortex transition.

We have also increased the event horizon radius r_h for several values of α and found no evidence of a maximal value of r_h where the solutions could eventually terminate. In figure 3 we plot a number of relevant physical quantities as a function of the event horizon radius for the special $d = 5$ and $d = 6$ cases (the corresponding plots for $d = 6, 7$ present the same qualitative features as the Einstein gravity data exhibited in [20] and we shall not present them here). (Note that all values of M , \mathcal{T} and S plotted in this section are divided by a factor $V_{k,d-2}$. Also, we have subtracted the Casimir terms from the $d = 5$ values of the mass and tension, as computed according to (4.15)).

We have also studied the dependence of the $k = 1$ solutions of the GB parameter α , for fixed values of the event horizon radius (see figure 4). The solutions exist up to a maximal value of α . The configuration corresponding to α_{\max} does not present any special properties. For those values of r_h where solutions were found, we strongly suspect the existence of a second branch of solutions, also terminating at $\alpha = \alpha_{\max}$, but it was not attempted to construct it in a systematic way.

Considering now topological black string solutions, our results show that their qualitative features are similar to those in the Einstein gravity. They also exist for all values of $r_h > r_{h(\min)}$, with $r_{h(\min)}$ decreasing with α . The mass, temperature and entropy of the $k = -1$ configurations increase monotonically with r_h , while the tension decreases (see figure 5). As a result, the thermodynamic of these solutions is similar to the $\alpha = 0$ case and they present a positive specific heat $C = T_H(\partial S/\partial T_H) > 0$, although, from (4.21) (and similar to the black hole case), the entropy may take also negative values.

However, when considering the thermodynamics of black strings, the situation is much

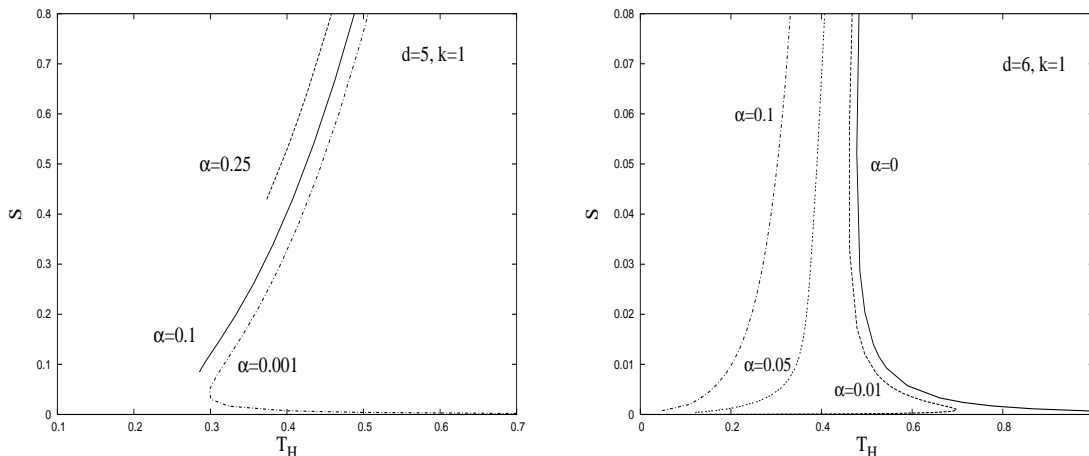


Figure 6: The entropy is plotted as a function of Hawking temperature for $d = 5, 6$ black string solutions with $k = 1$ and several different values of the parameter α .

more complicated for $k = 1$ solutions. Restricting to the case of local stability, the behaviour of configurations with $\alpha = 0$ follows the pattern of the corresponding black hole solutions in AdS background [20]. That is small black strings have negative specific heat (they are unstable) but large size black strings have positive specific heat (and they are stable). However, the picture for $d = 6$ is somehow similar to the one of the electrically charged black strings in the canonical ensemble discussed in [54]. Our results suggest also that for large enough values of the parameter α only one branch of stable solutions exist. As the coupling constant decreases below a critical value, additional branches appear, of intermediate and small sizes, of which the former has negative specific heat while the small black strings branch has positive specific heat (see figure 6). The picture for $d = 5$ is complicated by the existence of a minimal value of the event horizon radius for a given value of α . For small enough values of α , one finds two branches of solutions, with a negative specific heat for the small black strings branch. As seen in figure 6, if the GB parameter is large enough, all black strings solutions are locally stable. As expected, for $d > 6$, the thermodynamics of the EGB black strings exhibits the same qualitative features as in the Einstein gravity case.

5. Further remarks

The main purpose of this work was to present the boundary counterterm that removes the divergences of the action and conserved quantities of the solutions in EGB theory with negative cosmological constant for a spacetime dimension $d \leq 9$. The basic pieces are those used in Einstein gravity plus Lovelock gravity densities. Their coefficients, however, present an explicit dependence of the dimensionless factor $\alpha\Lambda$.

One expects that once these coefficients are fixed, one may use the same counterterms to regulate the action for any choice of coordinates on any asymptotically AdS solution in EGB theory. This approach is useful particularly for the cases where appropriate backgrounds are ambiguous or unknown. For example, we have found that the proposed

boundary counterterms provides a finite action and mass for the NUT-charged solutions in EGB gravity found in [66]. These configurations are particularly interesting, since as discussed *e.g.* in [67], the usual relationship between area and entropy is already violated in Einstein theory for a nonzero NUT charge. The boundary counterterm formalism provides a possibility to evaluate the entropy of such solutions in theories with higher derivatives.

The counterterm proposed in section 2 has been derived by considering a range of asymptotically AdS solutions in EGB gravity, which does not guarantee its universality. However, on general grounds, one expects the boundary counterterm action in EGB theory to be universal, being composed of a unique linear combination of curvature invariants that cancel the divergences in the total action in the limit when the boundary contains the full spacetime. For asymptotically AdS solutions in the Einstein gravity, there exist an algorithmic procedure for constructing I_{ct} in a rigorous way, and so its determination is unique for $\alpha = 0$ [68]. This procedure involves solving the Einstein equations (written in Gauss-Codacci form) in terms of the extrinsic curvature functional of the boundary and its derivatives to isolate the divergent parts. All divergent contributions can be expressed in terms of intrinsic boundary data and do not depend on normal derivatives. In principle, this approach can be extended to asymptotically AdS solutions in EGB theory, the only obstacle being the tremendous complexity of the required computation.

In the second part of our paper we have applied the general formalism to two different kind of asymptotically AdS black objects in EGB theory. The results we have found in the static black hole case are similar to those exhibited in the literature by employing a different approach. In section 4 we have presented a set of new solutions, generalizing the Einstein gravity black strings with $\Lambda < 0$ to EGB theory. As argued there, the presence of a GB term in the lagrangean leads to some interesting new features in five and six dimensions, in particular the absence of vortex-like solutions without an event horizon. The phase structure there gets also modified when including α -corrections. For example, in addition to the large thermodynamically stable black strings, there are also small $d = 6$ stable solutions along with intermediate unstable ones. The issue of Gregory-Laflamme instability of these black string is an interesting question. Based on the Gubser-Mitra conjecture [69] that correlates the dynamical and thermodynamical stability, we expect a more complex picture in this case than for the Einstein gravity solutions [58].

One can address also the situation when matter fields are added to the bulk action (2.1). One can verify that the counterterms proposed in section 2 regularize [70] the action and mass of the Reissner-Nordstrom generalizations of the black holes (3.1). However, the situation is different for $d \geq 5$ nonabelian theories [55, 71] or for $d = 5$ black strings with a magnetic U(1) field [56], in which case new, non-geometric counterterms should be added to the action already for $\alpha = 0$. An interesting open problem would be to find the corresponding counterterm expression for $d \geq 7$ asymptotically AdS solutions with higher order terms in the Lovelock gravity.

Acknowledgments

We would like to thank R. B. Mann for helpful remarks on a draft of this paper. Y. B. thanks the Belgian FNRS for financial support.

References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)].
- [2] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1;
R.R. Metsaev and A.A. Tseytlin, *Two loop β -function for the generalized bosonic σ -model*, *Phys. Lett.* **B 191** (1987) 354;
C.G. Callan Jr., R.C. Myers and M.J. Perry, *Black holes in string theory*, *Nucl. Phys.* **B 311** (1989) 673.
- [3] R.C. Myers, *Higher derivative gravity, surface terms and string theory*, *Phys. Rev.* **D 36** (1987) 392.
- [4] M.T. Grisaru and D. Zanon, *σ -model superstring corrections to the Einstein-Hilbert action*, *Phys. Lett.* **B 177** (1986) 347;
M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, *Higher order σ -model counterterms and the effective action for superstrings*, *Phys. Lett.* **B 178** (1986) 199;
A.A. Tseytlin, *Ambiguity in the effective action in string theories*, *Phys. Lett.* **B 176** (1986) 92.
- [5] A. Fayyazuddin and M. Spalinski, *Large- N superconformal gauge theories and supergravity orientifolds*, *Nucl. Phys.* **B 535** (1998) 219 [[hep-th/9805096](#)].
- [6] O. Aharony, A. Fayyazuddin and J.M. Maldacena, *The large- N limit of $N = 2, 1$ field theories from three-branes in F -theory*, *JHEP* **07** (1998) 013 [[hep-th/9806159](#)].
- [7] S. Nojiri and S.D. Odintsov, *Brane-world cosmology in higher derivative gravity or warped compactification in the next-to-leading order of AdS/CFT correspondence*, *JHEP* **07** (2000) 049 [[hep-th/0006232](#)].
- [8] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)].
- [9] J.D. Brown and J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev.* **D 47** (1993) 1407.
- [10] R.-G. Cai and N. Ohta, *Surface counterterms and boundary stress-energy tensors for asymptotically non-anti-de Sitter spaces*, *Phys. Rev.* **D 62** (2000) 024006 [[hep-th/9912013](#)].
- [11] G. Kofinas and R. Olea, *Vacuum energy in Einstein-Gauss-Bonnet AdS gravity*, *Phys. Rev.* **D 74** (2006) 084035 [[hep-th/0606253](#)].
- [12] O. Mišković and R. Olea, *Counterterms in dimensionally continued AdS gravity*, *JHEP* **10** (2007) 028 [[arXiv:0706.4460](#)].
- [13] G. Kofinas and R. Olea, *Universal regularization prescription for Lovelock AdS gravity*, *JHEP* **11** (2007) 069 [[arXiv:0708.0782](#)].
- [14] R. Olea, *Regularization of odd-dimensional AdS gravity: kounterterms*, *JHEP* **04** (2007) 073 [[hep-th/0610230](#)].
- [15] M. Cvetič, S. Nojiri and S.D. Odintsov, *Black hole thermodynamics and negative entropy in deSitter and anti-de Sitter Einstein-Gauss-Bonnet gravity*, *Nucl. Phys.* **B 628** (2002) 295 [[hep-th/0112045](#)].

- [16] S. Nojiri, S.D. Odintsov and S. Ogushi, *Cosmological and black hole brane world universes in higher derivative gravity*, *Phys. Rev. D* **65** (2002) 023521 [[hep-th/0108172](#)].
- [17] Y. Brihaye and E. Radu, *Five-dimensional rotating black holes in Einstein-Gauss-Bonnet theory*, *Phys. Lett. B* **661** (2008) 167 [[arXiv:0801.1021](#)].
- [18] M.H. Dehghani and R.B. Mann, *Thermodynamics of rotating charged black branes in third order Lovelock gravity and the counterterm method*, *Phys. Rev. D* **73** (2006) 104003 [[hep-th/0602243](#)];
M.H. Dehghani, N. Bostani and A. Sheikhi, *Counterterm method in Lovelock theory and horizonless solutions in dimensionally continued gravity*, *Phys. Rev. D* **73** (2006) 104013 [[hep-th/0603058](#)].
- [19] K. Copsey and G.T. Horowitz, *Gravity dual of gauge theory on $S^2 \times S^1 \times R$* , *JHEP* **06** (2006) 021 [[hep-th/0602003](#)].
- [20] R.B. Mann, E. Radu and C. Stelea, *Black string solutions with negative cosmological constant*, *JHEP* **09** (2006) 073 [[hep-th/0604205](#)].
- [21] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498.
- [22] A. Mardones and J. Zanelli, *Lovelock-Cartan theory of gravity*, *Class. and Quant. Grav.* **8** (1991) 1545.
- [23] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev. D* **15** (1977) 2752.
- [24] N. Okuyama and J.-I. Koga, *Asymptotically anti de Sitter spacetimes and conserved quantities in higher curvature gravitational theories*, *Phys. Rev. D* **71** (2005) 084009 [[hep-th/0501044](#)].
- [25] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 104001 [[hep-th/9903238](#)].
- [26] R.B. Mann, *Misner string entropy*, *Phys. Rev. D* **60** (1999) 104047 [[hep-th/9903229](#)].
- [27] S. Das and R.B. Mann, *Conserved quantities in Kerr-anti-de Sitter spacetimes in various dimensions*, *JHEP* **08** (2000) 033 [[hep-th/0008028](#)].
- [28] S.C. Davis, *Generalised Israel junction conditions for a Gauss-Bonnet brane world*, *Phys. Rev. D* **67** (2003) 024030 [[hep-th/0208205](#)];
E. Gravanis and S. Willison, *Israel conditions for the Gauss-Bonnet theory and the Friedmann equation on the brane universe*, *Phys. Lett. B* **562** (2003) 118 [[hep-th/0209076](#)].
- [29] F. Mueller-Hoissen, *Spontaneous compactification with quadratic and cubic curvature terms*, *Phys. Lett. B* **163** (1985) 106.
- [30] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, [hep-th/0404176](#).
- [31] R.C. Myers, *Stress tensors and Casimir energies in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 046002 [[hep-th/9903203](#)].
- [32] D. Astefanesei and E. Radu, *Quasilocal formalism and black ring thermodynamics*, *Phys. Rev. D* **73** (2006) 044014 [[hep-th/0509144](#)].

- [33] J.D. Brown, E.A. Martinez and J.W. York, *Complex Kerr-Newman geometry and black hole thermodynamics*, *Phys. Rev. Lett.* **66** (1991) 2281;
 J.D. Brown and J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev. D* **47** (1993) 1407 [[gr-qc/9209012](#)]; *The path integral formulation of gravitational thermodynamics*, *Phys. Rev. D* **47** (1993) 1420 [[gr-qc/9405024](#)];
 I.S. Booth and R.B. Mann, *Complex instantons and charged rotating black hole pair creation*, *Phys. Rev. Lett.* **81** (1998) 5052 [[gr-qc/9806015](#)]; *Cosmological pair production of charged and rotating black holes*, *Nucl. Phys. B* **539** (1999) 267 [[gr-qc/9806056](#)].
- [34] S.W. Hawking, *General relativity. An Einstein centenary survey*, S.W. Hawking and W. Israel eds., Cambridge University Press, Cambridge U.K. (1979).
- [35] R.B. Mann, *Expanding the area of gravitational entropy*, *Found. Phys.* **33** (2003) 65 [[gr-qc/0211047](#)].
- [36] D.G. Boulware and S. Deser, *String generated gravity models*, *Phys. Rev. Lett.* **55** (1985) 2656.
- [37] R.-G. Cai, *Gauss-Bonnet black holes in AdS spaces*, *Phys. Rev. D* **65** (2002) 084014 [[hep-th/0109133](#)].
- [38] S. Nojiri and S.D. Odintsov, *(Anti-) de Sitter black holes in higher derivative gravity and dual conformal field theories*, *Phys. Rev. D* **66** (2002) 044012 [[hep-th/0204112](#)].
- [39] Y.M. Cho and I.P. Neupane, *Anti-de Sitter black holes, thermal phase transition and holography in higher curvature gravity*, *Phys. Rev. D* **66** (2002) 024044 [[hep-th/0202140](#)];
 I.P. Neupane, *Thermodynamic and gravitational instability on hyperbolic spaces*, *Phys. Rev. D* **69** (2004) 084011 [[hep-th/0302132](#)]; *Black hole entropy in string-generated gravity models*, *Phys. Rev. D* **67** (2003) 061501 [[hep-th/0212092](#)].
- [40] S. Dutta and R. Gopakumar, *On Euclidean and Noetherian entropies in AdS space*, *Phys. Rev. D* **74** (2006) 044007 [[hep-th/0604070](#)].
- [41] T. Clunan, S.F. Ross and D.J. Smith, *On Gauss-Bonnet black hole entropy*, *Class. and Quant. Grav.* **21** (2004) 3447 [[gr-qc/0402044](#)].
- [42] A. Padilla, *Surface terms and the Gauss-Bonnet Hamiltonian*, *Class. and Quant. Grav.* **20** (2003) 3129 [[gr-qc/0303082](#)].
- [43] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev. D* **48** (1993) 3427 [[gr-qc/9307038](#)].
- [44] H.-C. Kim and R.-G. Cai, *Slowly rotating charged Gauss-Bonnet black holes in AdS spaces*, *Phys. Rev. D* **77** (2008) 024045 [[arXiv:0711.0885](#)].
- [45] M.H. Dehghani, *Charged rotating black branes in anti-de Sitter Einstein-Gauss-Bonnet gravity*, *Phys. Rev. D* **67** (2003) 064017 [[hep-th/0211191](#)]; *Magnetic branes in Gauss-Bonnet gravity*, *Phys. Rev. D* **69** (2004) 064024 [[hep-th/0312030](#)];
 M.H. Dehghani and R.B. Mann, *Thermodynamics of rotating charged black branes in third order Lovelock gravity and the counterterm method*, *Phys. Rev. D* **73** (2006) 104003 [[hep-th/0602243](#)];
 M.H. Dehghani, G.H. Bordbar and M. Shamirzaie, *Thermodynamics of rotating solutions in Gauss-Bonnet-Maxwell gravity and the counterterm method*, *Phys. Rev. D* **74** (2006) 064023 [[hep-th/0607067](#)].

- [46] J. Kunz, F. Navarro-Lerida and J. Viebahn, *Charged rotating black holes in odd dimensions*, *Phys. Lett. B* **639** (2006) 362 [[hep-th/0605075](#)].
- [47] G.W. Gibbons, H. Lu, D.N. Page and C.N. Pope, *Rotating black holes in higher dimensions with a cosmological constant*, *Phys. Rev. Lett.* **93** (2004) 171102 [[hep-th/0409155](#)].
- [48] N. Deruelle and Y. Morisawa, *Mass and angular momenta of Kerr anti-de Sitter spacetimes in Einstein-Gauss-Bonnet theory*, *Class. and Quant. Grav.* **22** (2005) 933 [[gr-qc/0411135](#)].
- [49] S. Deser, I. Kanik and B. Tekin, *Conserved charges of higher D Kerr-AdS spacetimes*, *Class. and Quant. Grav.* **22** (2005) 3383 [[gr-qc/0506057](#)].
- [50] G.T. Horowitz and A. Strominger, *Black strings and p-branes*, *Nucl. Phys. B* **360** (1991) 197.
- [51] A. Chamblin, S.W. Hawking and H.S. Reall, *Brane-world black holes*, *Phys. Rev. D* **61** (2000) 065007 [[hep-th/9909205](#)].
- [52] D. Kastor and R.B. Mann, *On black strings and branes in Lovelock gravity*, *JHEP* **04** (2006) 048 [[hep-th/0603168](#)].
- [53] A.H. Chamseddine and W.A. Sabra, *Magnetic strings in five dimensional gauged supergravity theories*, *Phys. Lett. B* **477** (2000) 329 [[hep-th/9911195](#)];
D. Klemm and W.A. Sabra, *Supersymmetry of black strings in D = 5 gauged supergravities*, *Phys. Rev. D* **62** (2000) 024003 [[hep-th/0001131](#)];
W.A. Sabra, *Magnetic branes in d-dimensional AdS Einstein-Maxwell gravity*, *Phys. Lett. B* **545** (2002) 175 [[hep-th/0207128](#)].
- [54] Y. Brihaye, E. Radu and C. Stelea, *Black strings with negative cosmological constant: inclusion of electric charge and rotation*, *Class. and Quant. Grav.* **24** (2007) 4839 [[hep-th/0703046](#)].
- [55] Y. Brihaye and E. Radu, *Magnetic solutions in AdS₅ and trace anomalies*, *Phys. Lett. B* **658** (2008) 164 [[arXiv:0706.4378](#)].
- [56] A. Bernamonti et al., *Black strings in AdS₅*, *JHEP* **01** (2008) 061 [[arXiv:0708.2402](#)].
- [57] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [[hep-th/9301052](#)].
- [58] Y. Brihaye, T. Delsate and E. Radu, *On the stability of AdS black strings*, *Phys. Lett. B* **662** (2008) 264 [[arXiv:0710.4034](#)].
- [59] T. Delsate, *Perturbative non uniform black strings in AdS₆*, *Phys. Lett. B* **663** (2008) 118 [[arXiv:0802.1392](#)].
- [60] T. Kobayashi and T. Tanaka, *Five-dimensional black strings in Einstein-Gauss-Bonnet gravity*, *Phys. Rev. D* **71** (2005) 084005 [[gr-qc/0412139](#)].
- [61] K. Skenderis, *Asymptotically anti-de Sitter spacetimes and their stress energy tensor*, *Int. J. Mod. Phys. A* **16** (2001) 740 [[hep-th/0010138](#)];
M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)]; *Holography and the Weyl anomaly*, *Fortschr. Phys.* **48** (2000) 125 [[hep-th/9812032](#)].
- [62] P.K. Townsend and M. Zamaklar, *The first law of black brane mechanics*, *Class. and Quant. Grav.* **18** (2001) 5269 [[hep-th/0107228](#)].

- [63] H. Kudoh and T. Wiseman, *Connecting black holes and black strings*, *Phys. Rev. Lett.* **94** (2005) 161102 [[hep-th/0409111](#)];
B. Kleihaus, J. Kunz and E. Radu, *New nonuniform black string solutions*, *JHEP* **06** (2006) 016 [[hep-th/0603119](#)].
- [64] F. Bastianelli, S. Frolov and A.A. Tseytlin, *Conformal anomaly of (2,0) tensor multiplet in six dimensions and AdS/CFT correspondence*, *JHEP* **02** (2000) 013 [[hep-th/0001041](#)].
- [65] U. Ascher, J. Christiansen and R.D. Russell, *A collocation solver for mixed order systems of boundary value problems*, *Math. Comp.* **33** (1979) 659; *Collocation software for boundary-value ODEs*, *ACM Trans. Math. Software* **7** (1981) 209.
- [66] M.H. Dehghani and R.B. Mann, *NUT-charged black holes in Gauss-Bonnet gravity*, *Phys. Rev. D* **72** (2005) 124006 [[hep-th/0510083](#)];
M.H. Dehghani and S.H. Hendi, *Taub-NUT/Bolt black holes in Gauss-Bonnet-Maxwell gravity*, *Phys. Rev. D* **73** (2006) 084021 [[hep-th/0602069](#)].
- [67] D. Astefanesei, R.B. Mann and E. Radu, *Breakdown of the entropy/area relationship for NUT-charged spacetimes*, *Phys. Lett. B* **620** (2005) 1 [[hep-th/0406050](#)]; *Nut charged space-times and closed timelike curves on the boundary*, *JHEP* **01** (2005) 049 [[hep-th/0407110](#)].
- [68] P. Kraus, F. Larsen and R. Siebelink, *The gravitational action in asymptotically AdS and flat spacetimes*, *Nucl. Phys. B* **563** (1999) 259 [[hep-th/9906127](#)].
- [69] S.S. Gubser and I. Mitra, *Instability of charged black holes in anti-de Sitter space*, [hep-th/0009126](#); *The evolution of unstable black holes in anti-de Sitter space*, *JHEP* **08** (2001) 018 [[hep-th/0011127](#)].
- [70] D. Astefanesei, N. Banerjee and S. Dutta, *(Un)attractor black holes in higher derivative AdS gravity*, [arXiv:0806.1334](#).
- [71] E. Radu and D.H. Tchrakian, *No hair conjecture, nonabelian hierarchies and anti-de Sitter spacetime*, *Phys. Rev. D* **73** (2006) 024006 [[gr-qc/0508033](#)].